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PARTIAL REGULARITY OF SOLUTIONS TO A CLASS OF DEGENERATE SYSTEMS

XIANGSHENG XU

ABSTRACT. We consider the system $\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2$, $\operatorname{div}(\sigma(u)\nabla\varphi) = 0$ in $Q_T \equiv \Omega \times (0,T]$ coupled with suitable initial-boundary conditions, where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary and $\sigma(u)$ is a continuous and positive function of u. Our main result is that under some conditions on σ there exists a relatively open subset Q_0 of Q_T such that u is locally Hölder continuous on Q_0, the interior of $Q_T \backslash Q_0$ is empty, and u is essentially bounded on $Q_T \backslash Q_0$.

1. Introduction

In this paper we obtain several results concerning the regularity of solutions to the system

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\varphi|^2 \qquad in \qquad Q_T \equiv \Omega \times (0,T], \tag{1.1a}$$

$$\operatorname{div}(\sigma(u)\nabla\varphi) = 0 \qquad in \qquad Q_T\ , \tag{1.1b}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $T > 0$, and $\sigma \in C(\mathbf{R})$ is positive. In [SSX], this system is proposed as a model for a conductor in which both heat conduction and electrical conduction take place. Then u is the temperature of the conductor, and φ the electrical potential. The first equation describes the diffusion of heat, while the second equation represents the conservation of electrical charges. The term $\sigma(u)$ is the temperature-dependent electrical conductivity. Its precise form is determined by the particular physical situation one has in mind. See [XA] for the expression for $\sigma(u)$ in the microsensor applications.

To complete the problem, the system needs to be coupled with suitable initial-boundary conditions. Here we impose the following conditions:

$$u = \overline{u} \qquad on \qquad S_T \equiv \partial\Omega \times (0,T], \tag{1.1c}$$

$$\varphi = \overline{\varphi} \qquad on \qquad S_T, \tag{1.1d}$$

$$u(x,0) = u_0(x) \qquad on \qquad \Omega. \tag{1.1e}$$

Here, $\overline{u}$, $\overline{\varphi}$, u_0 satisfy certain conditions to be specified later.

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We are interested in the question of how different assumptions on σ may affect the regularity of solutions of the problem. If $\sigma \in C^1(\mathbf{R})$ is such that

$$m \leq \sigma(s) \leq M \qquad on \qquad \mathbf{R}$$

for some $m, M \in (0,\infty)$, a result of [YL, Y] indicates that $u \in C^{\alpha,\frac{\alpha}{2}}\left(\overline{Q_T}\right)$ for some $\alpha \in (0,1)$. This immediately triggers a bootstrap process which yields higher regularity for u and φ if σ is smooth enough. The proof in [YL] relies upon the single-layer potential theory. As a by-product of our development here we give a new proof of this result which seems to exhibit a new perspective from which to view the system. We are mainly concerned with the case where σ is bounded above and satisfies

$$\lim_{\tau\to 0}\frac{\sigma(s+\tau)}{\sigma(s)} = 1 \qquad \text{uniformly on} \quad \mathbf{R}, \tag{1.2}$$

and

$$\lim_{s\to\infty}\sigma(s) = 0. \tag{1.3}$$

The conditions here are rather general [C]. For example, exponential functions of the form $\sigma(s) = e^{-\alpha|s|}$, $\alpha > 0$, satisfy (1.2) and (1.3). In this generality, we do not expect u to be bounded. If u is unbounded, the system degenerates on the set where $|u|$ is infinite. We establish that there exists a relatively open subset Q_0 of Q_T such that u is locally Hölder continuous on Q_0, the interior of $Q_T\backslash Q_0$ is empty, and $u \in L^\infty(Q_T\backslash Q_0)$. Thus $u(x,t)$ may become unbounded only when $(x,t) \in Q_0$ approaches the set $Q_T\backslash Q_0$, the boundary of Q_0.

Generally speaking, the study of systems of partial differential equations consists of two steps [K]: first one proves the existence of a solution in a suitable function space; then one proves the regularity of this solution. For some systems, the second step may be false (see [G, Chap. 2]), and the solutions may have singularities. In such cases, one is forced to seek a partial regularity theorem, giving a description of the set of possible singularities. This approach has been used with great success in a variety of settings (see [K] and the references therein). This paper contains the first partial regularity results for models of the electrical heating of conductors.

Our method relies upon the following weak Harnack inequality:

$$\fint_{Q_{2R}(x,t)} u\,dy\,d\tau \leq \operatorname{ess}\inf_{Q_R(x,t)} u + c, \tag{1.4}$$

where $R > 0, Q_R(x,t) \equiv \left\{(y,\tau) : |y-x| < R \text{ and } t - R^2 < \tau \leq t\right\}$, and c is a positive constant. This inequality is a consequence of the following two facts: First, u is a supersolution of the classical heat equation $\frac{\partial u}{\partial t} - \Delta u = 0$. Second, $u \in BMO$ [FS] in the parabolic sense. To be precise, we have

$$\fint_{Q_R(x,t)} (u - u_R)^2\,dy\,d\tau \leq c, \tag{1.5}$$

where

$$u_R = \fint_{Q_R(x,t)} u\,dy\,d\tau.$$

Section 2 is devoted to the proof of (1.4) and (1.5). Our main result is established in Section 3.

Our system consists of an equation of parabolic type and an equation of elliptic type. Thus, there is no clear theoretical framework available to handle the situation. What saves the game here is a clever combination of some elliptic techniques and some parabolic techniques. Since the second equation does not involve the term $\frac{\partial \varphi}{\partial t}$, we are not able to obtain the Hausdorff dimension of $Q_T \backslash Q_0$. In this respect, our partial regularity result is different from that in [G]. However, we should be able to remove (1.3). We wish to consider this possibility in a forthcoming paper. For other related work, we refer the reader to [C, AC] and references therein.

Finally, we make some remarks about the notation. For $R > 0$, $(x_0, t_0) \in \mathbf{R}^{N+1}$, let

$$\begin{aligned}
B_R(x_0) &= \{x : |x - x_0| < R\}, \\
\Omega_R(x_0) &= \Omega \cap B_R(x_0), \\
Q_R(x_0, t_0) &= B_R(x_0) \times \left(t_0 - R^2, t_0\right], \\
D_R(x_0, t_0) &= \Omega_R(x_0) \times \left(t_0 - R^2, t_0\right], \\
P_R(x_0, t_0) &= Q_R(x_0, t_0) \cap Q_T, \\
\partial p P_R(x_0, t_0) &= \text{the parabolic boundary of } P_R(x_0, t_0).
\end{aligned}$$

The letter c, or c_i, $i \in \{0, 1, \dots\}$, is used to denote a generic positive constant. In the expression

$$\int_{P_R(x,t)} (u - u_R)^2 \, dy d\tau,$$

we defne

$$u_R = \fint_{P_R(x,t)} u dy d\tau.$$

For $(x, t) \in \overline{Q}_T$, define

$$\begin{aligned}
R_*(x) &= \operatorname{dist}(x, \partial\Omega), \\
R_0(x, t) &= \min\left\{R_*(x), \sqrt{\min\left\{(R_*(x))^2, t\right\}}\right\}.
\end{aligned}$$

Then

$$Q_{R_0(x,t)}(x, t) \subset Q_T$$

and

$$Q_R(x, t) \cap \left(\mathbf{R}^{N+1} \backslash Q_T\right) \neq \emptyset$$

for each $R > R_0(x, t)$.

2. A weak Harnack inequality for u

We first list our assumptions on the data.

(H1) $\sigma \in C(\mathbf{R})$ is such that

$$0 < \sigma \leq M \qquad on \qquad \mathbf{R}$$

for some $M \in (0, \infty)$.

(H2) There exists a function u^* in $W^{1,\infty}(Q_T)$ with the properties

$$\begin{aligned}
u^* &= \overline{u} \qquad on \qquad S_T, \\
u^* &= u_0 \qquad on \qquad \Omega \times \{0\}.
\end{aligned}$$

(H3) $\overline{\varphi} \in L^\infty\left(0, T;\ W^{1,\infty}(\Omega)\right)$.

We remark that most of our subsequent results require weaker assumptions. For simplicity we will operate under (H1)–(H3) throughout the paper.

In the generality considered here it is not clear to us a priori that u is bounded. As a result, the equation (1.1b) may degenerate. This means that we will not be able to obtain an estimate of the type $|\nabla\varphi| \in L^p(Q_T)$, $p \geq 1$. To solve the question of in what sense the system is satisfied, we appeal to the notion of a capacity solution introduced in [X1].

Definition 1. A triplet $\{u, \varphi, g\}$ is said to be a capacity solution of (1.1) if:

(i) $u - u^* \in C([0,T]; L^2(\Omega)) \cap L^2\left(0,T;\ W_0^{1,2}(\Omega)\right)$, $\varphi \in L^\infty(Q_T)$, $g \in \left[L^2(Q_T)\right]^N$;

(ii)

$$\begin{array}{c} \frac{\partial u}{\partial t} - \Delta u = \operatorname{div}(\varphi g) \qquad in \qquad L^2\left(0,T;\ W^{-1,2}(\Omega)\right), \\ \operatorname{div} g = 0 \qquad in \qquad L^2\left(0,T;\ W^{-1,2}(\Omega)\right); \end{array}$$

(iii) for each $\theta \in C_0^1(\mathbf{R})$, $\theta(u)\varphi - \theta(u^*)\overline{\varphi} \in L^2\left(0,T;\ W_0^{1,2}(\Omega)\right)$ and $\theta(u) g = \sigma(u)(\nabla(\theta(u)\varphi) - \varphi\nabla\theta(u))$;

(iv) $u(x,0) = u_0(x)$ in $L^2(\Omega)$.

A detailed analysis of this definition is presented in [X1]. Here we only point out that (iii) implies $g = \sigma(u)\nabla\varphi$, where $\nabla\varphi$ is a measurable function defined as in [X1]. A result of [X1] asserts that under (H1)–(H3), problem (1.1) has a capacity solution.

We are ready to state the main result of this section.

Theorem 2. *Let $\{u, \varphi\}$ be a capacity solution of (1.1). Then there exists a $c^* > 0$ depending only on the data such that*

$$\fint_{Q_{2R}(x_0,t_0)} u(y,\tau)\,dy d\tau \leq ess \inf_{Q_R(x_0,t_0)} u + c^* \tag{2.1}$$

for all $(x_0, t_0) \in Q_T$, $R > 0$ such that $2R \leq R_0(x_0, t_0)$.

Before we continue, let us cite the following known result.

Lemma 3 (Chain rule)**.** *Let $v \in L^2\left(0,T;\ W_0^{1,2}(\Omega)\right)$ be such that*

$$v_t \in L^2\left(0,T;\ W^{-1,2}(\Omega)\right).$$

Then for any Lipschitz function θ with $\theta(0) = 0$, the function $t \to \int_\Omega \int_0^{u(x,t)} \theta(s)\,ds dx$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\int_\Omega \int_0^{u(x,t)} \theta(s)\,ds dx = (u_t, \theta(u)) \qquad a.e. \quad on \quad (0,T),$$

where $(\cdot,\cdot)$ denotes the duality pairing between $W^{-1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$.

The core of the proof of Theorem 2 is contained in the following lemmas.

Lemma 4. *There exists $c > 0$ such that*

$$\int_{P_R(x_0,t_0)} \sigma(u)\,|\nabla\varphi|^2\,dx dt \leq C R^N$$

for all $R > 0$ and $(x_0, t_0) \in \overline{Q}_T$.

Proof. First observe from (ii) that

$$\operatorname{div}\left(\sigma\left(u\right)\nabla\varphi\right)=0 \qquad in \qquad W^{-1,2}\left(\Omega\right) \tag{2.2}$$

for a.e. $t\in(0,T)$. Now let $(x_0,t_0)\in\overline{Q}_T$, $R>0$ be given. Set $B_R=B_R\left(x_0\right)$. Choose $\xi\in C_0^\infty\left(B_{2R}\right)$ so that $\xi=1$ on B_R, $\xi\geq 0$ on B_{2R}, and $|\nabla\xi|\leq 2/R$. For each $k\in\{1,2,\dots\}$, let $\theta_k\in C_0^1\left(\mathbf{R}\right)$ be such that

$$\begin{gathered}|\theta_k'|\leq\frac{c}{k},\\ \lim_{k\to\infty}\theta_k\left(s\right)=1 \text{ for all } s\in\mathbf{R},\\ \theta_k\geq 0 \qquad on \qquad \mathbf{R}.\end{gathered}$$

By (iii), $\theta_k\left(u\right)\varphi-\theta_k\left(u^*\right)\overline{\varphi}\in W_0^{1,2}\left(\Omega\right)$ for a.e. $t\in(0,T)$. As a test function in (2.2) we can use $\left(\theta_k\left(u\right)\varphi-\theta_k\left(u^*\right)\overline{\varphi}\right)\xi^2$ to obtain, for a.e. t in $(0,T)$, that

$$\begin{aligned}&\int_{\Omega_{2R}}\sigma\left(u\right)\nabla\varphi\nabla\left(\theta_k\left(u\right)\varphi\right)\xi^2dx\\ &\qquad=-\int_{\Omega_{2R}}\sigma\left(u\right)\nabla\varphi\theta_k\left(u\right)\varphi 2\xi\nabla\xi dx\\ &\qquad\quad+\int_{\Omega_{2R}}\sigma\left(u\right)\nabla\varphi\nabla\left(\theta_k\left(u^*\right)\overline{\varphi}\xi^2\right)dx.\end{aligned}$$

In light of (iii), we get, for a.e. t in $(0,T)$, that

$$\begin{aligned}&\int_{\Omega_{2R}}\sigma\left(u\right)\left|\nabla\varphi\right|^2\theta_k\left(u\right)\xi^2dx\\ &\qquad=-\int_{\Omega_{2R}}\sigma\left(u\right)\nabla\varphi\varphi\theta_k'\left(u\right)\nabla u\xi^2dx-\int_{\Omega_{2R}}\sigma\left(u\right)\nabla\varphi\theta_k\left(u\right)\varphi 2\xi\nabla\xi dx\\ &\qquad\quad+\int_{\Omega_{2R}}\sigma\left(u\right)\nabla\varphi\nabla\left(\theta_k\left(u^*\right)\overline{\varphi}\xi^2\right)dx.\end{aligned}$$

Take $k\to\infty$, apply Hölder's inequality in the resulting equation, keep in mind (H1)–(H3) and the fact that $\varphi\in L^\infty\left(Q_T\right)\equiv L^\infty\left(0,T;\,L^\infty\left(\Omega\right)\right)$, and thereby obtain

$$\int_{\Omega_R}\sigma\left(u\right)\left|\nabla\varphi\right|^2dx\leq CR^{N-2}$$

for a.e. t in $(0,T)$. Integration with respect to t yields the desired result. □

Lemma 5. *There exists $c>0$ such that for all $(x_0,t_0)\in Q_T$ and $R>0$ with $2R\leq R_0\left(x_0,t_0\right)$,*

$$⨍_{Q_{2R}(x_0,t_0)}udxdt\leq\operatorname{ess}\inf_{Q_R(x_0,t_0)}u+c\left(⨍_{Q_{2R}(x_0,t_0)}\left(u-u_{2R}\right)^2dxdt\right)^{\frac{1}{2}}. \tag{2.3}$$

Proof. Let $\{\theta_k\}$ be given as before. For $\xi \in L^2\left(0,T;\ W_0^{1,2}(\Omega)\right) \cap L^\infty(Q_T)$ and τ in $(0,T)$, we compute, with the aid of (ii) and (iii), that

$$\begin{aligned}
&\int_0^\tau \left(\operatorname{div}\left(\sigma(u)\nabla\varphi\varphi\right), \theta_k(u)\xi\right) dt \\
&\qquad = -\int_0^\tau\int_\Omega \sigma(u)\nabla\varphi\varphi\nabla\theta_k(u)\xi dxdt \\
&\qquad\quad -\int_0^\tau\int_\Omega \sigma(u)\nabla\varphi\varphi\theta_k(u)\nabla\xi dxdt \\
&\qquad = -\int_0^\tau\int_\Omega \sigma(u)\nabla\varphi\varphi\nabla\theta_k(u)\xi dxdt \\
&\qquad\quad -\int_0^\tau\int_\Omega \sigma(u)\nabla\varphi\left(\nabla\left(\xi\varphi\theta_k(u)\right) - \xi\nabla\left(\varphi\theta_k(u)\right)\right) dxdt \\
&\qquad = \int_0^\tau\int_\Omega \sigma(u)\left|\nabla\varphi\right|^2\theta_k(u)\xi dxdt.
\end{aligned}$$

Letting $k \to \infty$ yields

$$\int_0^\tau \left(\operatorname{div}\left(\sigma(u)\varphi\nabla\varphi\right), \xi\right) dt = \int_0^\tau\int_\Omega \sigma(u)\left|\nabla\varphi\right|^2\xi dxdt \tag{2.4}$$

for all $\xi \in L^2\left(0,T;\ W_0^{1,2}(\Omega)\right) \cap L^\infty(Q_T)$. Now fix $(x_0,t_0) \in Q_T$, $0 < 2R \leq R_0(x_0,t_0)$. Define, for $\varepsilon > 0$,

$$\eta_\varepsilon(s) = \begin{cases} 1 & if \quad s \leq 0, \\ -\frac{1}{\varepsilon}(s-\varepsilon) & if \quad 0 < s < \varepsilon, \\ 0 & if \quad s > \varepsilon. \end{cases}$$

Then, for any $A \in \mathbf{R}$, $\xi \in C_0^\infty\left(\mathbf{R}^{N+1}\right)$ with $\xi = 0$ on ∂pQ_T, we have $\eta_\varepsilon(u-A)\xi \in L^2\left(0,T;\ W_0^{1,2}(\Omega)\right)$. For such A, ξ we deduce from (ii) and (2.4) that for each $\tau \in (0,T]$,

$$\begin{aligned}
&\int_\Omega\int_0^{u-A}\eta_\varepsilon(s)ds\xi dx - \int_0^\tau\int_\Omega\int_0^{u-A}\eta_\varepsilon(s)ds\xi_t dxdt + \int_0^\tau\int_\Omega \nabla u\theta_\varepsilon(u-A)\nabla\xi dxdt \\
&\quad = -\int_0^\tau\int_\Omega \left|\nabla u\right|^2\eta_\varepsilon'(u-A)\xi dxdt + \int_0^\tau\int_\Omega \sigma(u)\left|\nabla\varphi\right|^2\eta_\varepsilon(u-A)\xi dxdt.
\end{aligned}$$

This implies that $(u-A)^-$ is a subsolution of the classical heat equation $\frac{\partial u}{\partial t}-\Delta u=0$. That is to say,

$$\int_0^\tau \left(\frac{\partial}{\partial t}(u-A)^- - \Delta (u-A)^-, \xi\right) dt \le 0$$

for all $\tau \in (0,T]$ and $\xi \in L^2\left(0,T;\ W_0^{1,2}(\Omega)\right)$ with $\xi = 0$ on $\partial p Q_T$. This asserts that there exists $c = c(N) > 0$ with

$$\operatorname*{ess\ sup}_{Q_R(x_0,t_0)} (u-A)^- \le c\left(\fint_{Q_{2R}} \left[(u-A)^-\right]^2 dxdt\right)^{\frac{1}{2}}.$$

We take $A = \fint_{Q_{2R}(x_0,t_0)} udxdt$ and obtain

$$\fint_{Q_{2R}(x_0,t_0)} udxdt \le \operatorname*{ess\ inf}_{Q_R(x_0,t_0)} u + c\left(\fint_{Q_{2R}(x_0,t_0)} \left[(u-u_{2R})^-\right]^2 dxdt\right)^{\frac{1}{2}}.$$

This completes the proof. □

In view of Lemma 5, we can obtain Theorem 2 by showing that

$$\fint_{Q_{2R}(x_0,t_0)} (u-u_{2R})^2\, dxdt \le c \tag{2.5}$$

for all $(x_0,t_0) \in Q_T$ and all $0 < R$ such that $2R \le R_0(x_0,t_0)$. To this end, set

$$z(x,t) = \begin{cases} u(x,t) - u^*(x,t) & if \quad (x,t) \in \overline{Q}_T\ , \\ 0 & if \quad t<0 \ and\ x \in \Omega. \end{cases}$$

Then one can easily check that z is the weak solution of the following problem

$$\frac{\partial}{\partial t} z - \Delta z = f \quad in \quad \Omega \times (-\tau, T], \tag{2.6a}$$

$$z = 0 \qquad on \qquad \partial p\left(\Omega \times (-\tau, T]\right), \tag{2.6b}$$

where $\tau =$ the diameter of Ω and $f = \operatorname{div}\left(\sigma(u)\varphi\nabla\varphi\chi_{Q_T} + \nabla u^*\chi_{Q_T}\right) - u_t^*\chi_{Q_T}$. Given that $(x_0,t_0) \in \overline{Q}_T \backslash \Omega \times \{0\}$ and $0 < R \le$ the diameter of Ω, consider the problem

$$\frac{\partial v}{\partial t} - \Delta v = f \qquad in \qquad D_R, \tag{2.7a}$$

$$v = 0 \qquad on \qquad \partial p D_R, \tag{2.7b}$$

where $D_R = D_R(x_0,t_0)$. We can easily conclude from [LSU] that this problem admits a unique solution in

$$C\left(\left[t_0-R^2,t_0\right];\ L^2(\Omega_R)\right) \cap L^2\left(\left(t_0-R^2,t_0\right);\ W_0^{1,2}(\Omega_R)\right),$$

where we write Ω_R for $\Omega_R(x_0)$. Use v as a test function in (2.7a) to obtain

$$\begin{aligned} |v|_{D_R} &\equiv \max_{[t_0-R^2,t_0]} \int_{\Omega_R} v^2(x,t)\,dx + \int_{D_R} |\nabla v|^2\, dxdt \\ &\le c\int_{P_R} \sigma(u)\,|\nabla\varphi|^2\, dxdt + cR^{N+2}. \end{aligned} \tag{2.8}$$

For convenience, assume $N \geq 3$. Then an embedding lemma in [D, p.7] indicates that

$$\|v\|^2_{2+\frac{4}{N},D_R} \leq c\,|v|_{D_R}\ .$$

Consequently,

$$\begin{aligned}\int\limits_{D_R} v^2 dxdt &\leq cR^2 \left(\int\limits_{D_R} |v|^{2+\frac{4}{N}}\, dxdt\right)^{\frac{N}{N+2}} \\ &\leq cR^2 \int\limits_{P_R} \sigma(u)\,|\nabla\varphi|^2\, dxdt \\ &\leq cR^{N+2} + CR^{N+4} \leq c_1 R^{N+2}.\end{aligned} \tag{2.9}$$

The last step is due to **Lemma 4**.

Let $w = u - v$. Then w is the weak solution of the following problem:

$$\frac{\partial w}{\partial t} - \Delta w = 0 \qquad in \qquad D_R, \tag{2.10a}$$

$$w = u \qquad in \qquad \partial p D_R. \tag{2.10b}$$

Claim 1. *If $D_R \subset Q_T$, i.e., $D_R = Q_R$, then there exist $c > 0$, $\alpha \in (0,1)$ depending only on N such that*

$$\int\limits_{Q_\rho(x_0,t_0)} (w - w_\rho)^2\, dxdt \leq c\left(\frac{\rho}{R}\right)^{N+2+2\alpha} \int\limits_{Q_R(x_0,t_0)} (w - w_R)^2\, dxdt \tag{2.11}$$

for all $0 < \rho \leq R$.

We first show that (2.11) holds for $\rho \leq \dfrac{R}{4}$. To do this, we appeal to a result in [D, p.42] which states that there exist $\alpha \in (0,1)$ and $c > 0$, which can be determined a priori only in terms of N, such that

$$|w(x_1,t_1) - w(x_2,t_2)| \leq c\,\|w - w_R\|_{\infty,Q_{\frac{R}{2}}} \left(\frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}}{R}\right)^{\alpha}$$

for all $(x_1,t_1), (x_2,t_2) \in Q_\rho$. Invoking another result in [D, p.123], we obtain that

$$\|w - w_R\|_{\infty,Q_{\frac{R}{2}}} \leq c(N)\left(⨍_{Q_R} (w - w_R)^2\, dxdt\right)^{\frac{1}{2}}.$$

Thus, if $0 < \rho \leq \dfrac{R}{4}$, we have that

$$\begin{aligned}\int\limits_{Q_\rho} |w - w_\rho|^2\, dxdt &\leq c_1 \|w - w_R\|^2_{\infty,Q_{\frac{R}{2}}} \left(\frac{\rho}{R}\right)^{2\alpha} \cdot \rho^{N+2} \\ &\leq c_1 ⨍_{Q_R} (w - w_R)^2\, dxdt \left(\frac{\rho}{R}\right)^{2\alpha} \rho^{N+2} \\ &= c_1 \left(\frac{\rho}{R}\right)^{N+2+2\alpha} \int\limits_{Q_R} (w - w_R)^2\, dxdt.\end{aligned}$$

If $\rho > \frac{R}{4}$, then $\frac{4\rho}{R} > 1$. Consequently,

$$\begin{aligned}\int_{Q_\rho} (w - w_\rho)^2 \, dxdt &\le \int_{Q_R} (w - w_R)^2 \, dxdt \\ &\le 4^{2+N+2\alpha} \left(\frac{\rho}{R}\right)^{2+N+2\alpha} \int_{Q_R} (w - w_R)^2 \, dxdt.\end{aligned} \tag{2.12}$$

Then (2.11) holds for $c = \max\left\{c_1, 4^{2+N+2\alpha}\right\}$.

We are ready to prove the following lemma.

Lemma 6. *There exist $c_1 > 0$, $c_2 > 0$ depending only on the data such that for all $(x_0, t_0) \in Q_T$ and $0 < \rho \le R_* = R_*(x_0)$,*

$$⨍_{Q_\rho(x_0,t_0)} (z - z_\rho)^2 \, dxdt \le c_1 ⨍_{Q_{R_*}(x_0,t_0)} (z - z_{R_*})^2 \, dxdt + c_2.$$

Proof. Fix (x_0, t_0) in Q_T and $0 < R \le R_*$. Let v, w be given as before. Then for each $0 < \rho \le R$, we calculate, with the aid of (2.9) and (2.11), that

$$\begin{aligned}&\int_{Q_\rho} (z - z_\rho)^2 \, dxdt \\ &\le c \int_{Q_\rho} (z - w)^2 \, dxdt + c \int_{Q_\rho} (w - w_\rho)^2 \, dxdt \\ &\le c \int_{Q_R} v^2 dxdt + c \left(\frac{\rho}{R}\right)^{N+2+2\alpha} \int_{Q_R} (w - w_R)^2 \, dxdt \\ &\le c \left(1 + \left(\frac{\rho}{R}\right)^{N+2+2\alpha}\right) \int_{Q_R} v^2 dxdt + c \left(\frac{\rho}{R}\right)^{N+2+2\alpha} \int_{Q_R} (z - z_R)^2 \, dxdt \\ &\le cR^{N+2} + c \left(\frac{\rho}{R}\right)^{N+2+2\alpha} \int_{Q_R} (z - z_R)^2 \, dxdt.\end{aligned} \tag{2.13}$$

In fact, this holds for all $0 < \rho \le R \le R_*$. Thus, we are in a position to apply Lemma 2.1 in [G, p.86]. This yields

$$⨍_{Q_\rho} (z - z_\rho)^2 \, dxdt \le c_1 ⨍_{Q_R} (z - z_R)^2 \, dxdt + c_2, \tag{2.14}$$

for all $0 \le \rho \le R \le R_*$. Set $R = R_*$ to obtain the desired result.

If $(x_0, t_0) \in Q_T$ is such that $R_0(x_0, t_0) \le R_*(x_0)$, we have, for all $0 < \rho \le R_0(x_0, t_0)$, that

$$\begin{aligned}⨍_{Q_\rho} (u - u_\rho)^2 \, dxdt &\le c ⨍_{Q_\rho} (z - z_\rho)^2 + c ⨍_{Q_\rho} \left(u^* - (u^*)_\rho\right)^2 dxdt \\ &\le c ⨍_{Q_{R_*}} (z - z_\rho)^2 \, dx + c \\ &\le c ⨍_{Q_{R^*}} z^2 dx + c.\end{aligned} \tag{2.15}$$

This implies that for all $(x,t) \in Q_T$ such that $\operatorname{dist}(x, \partial\Omega) \geq d > 0$, there exists a $c = c(d)$ with

$$\fint_{Q_\rho(x,t)} (u - u_\rho)^2 \, dxdt \leq c \fint_{Q_{R_0}(x,t)} (u - u_{R_0})^2 \, dxdt + c \\ \leq c(d)$$

for all $0 < \rho \leq R_0(x,t)$. To finish the proof of (2.5), we still have to show that for each $(x_0, t_0) \in S_T$, there is a $\rho_0 > 0$ such that $\fint_{Q_{R_0}(x,t)} (u - u_{R_0})^2 \, dyd\tau$ is bounded in $D_{\rho_0}(x_0, t_0)$. The idea here is to introduce a change of variables which flattens the relevant portion of the boundary $\partial\Omega$. Fix $x_0 \in \partial\Omega$. Since Ω is C^1, we can find an open neighborhood U about x_0, a number $A_0 > 0$, a C^1-homeomorphism $\mathcal{T}$ between $\Lambda \equiv \left\{ y \in \mathbf{R}^N : |y_i| < A_0, i = 1, \ldots, N \right\}$ and U such that

$$\begin{array}{lcl} \mathcal{T}^{-1}(U \cap \Omega) & = & \Lambda^+ \equiv \{ y \in \Lambda : 0 < y_N \}, \\ \mathcal{T}^{-1}\left(U \cap \left(\mathbf{R}^N \backslash \overline{\Omega}\right)\right) & = & \Lambda^- \equiv \{ y \in \Lambda : y_N < 0 \}, \\ \mathcal{T}^{-1}(x_0) & = & 0, \\ \mathcal{T}^{-1}(U \cap \partial\Omega) & = & \{ y \in \Lambda : y_N = 0 \}. \end{array}$$

Let $\mathcal{T}$ be defined by

$$x_i = g_i(y_1, \ldots, y_N), \qquad i = 1, \ldots, N.$$

Then set

$$H = \left(\frac{\partial}{\partial y_j} g_i \right).$$

Following the construction of $\mathcal{T}$ in [SW], we can choose g_i so that

$$\text{the Jacobian of } \mathcal{T} \equiv |\det H| = 1. \tag{2.16}$$

For $y \in \Lambda^+$, define

$$\widetilde{z}(y,t) = z\left(g_1(y_1, \ldots y_N), \cdots, g_N(y_1, \ldots, y_N), t\right).$$

An elementary calculation shows

$$\frac{\partial}{\partial t}\widetilde{z} - \operatorname{div}\left(\left(H^{-1}\right)^T H^{-1} \nabla \widetilde{z} \right) = \operatorname{div} g + g_0 \qquad in \qquad \Lambda^+ \times (-\tau, T], \tag{2.17}$$

where

$$\begin{array}{lcl} g(y,t) & = & H^{-1}\left(\sigma(u)\varphi\nabla\varphi\chi_{Q_T} + \nabla u^* \chi_{Q_T}\right)\big|_{x=\mathcal{T}y}, \\ g_0(y,t) & = & -u_t^* \, \chi_{Q_T}\big|_{x=\mathcal{T}y}, \\ \left(H^{-1}\right)^T H^{-1} & = & \left(H^{-1}\right)^T H^{-1}\Big|_{x=\mathcal{T}y}. \end{array}$$

Note that in the derivation of (2.17) we made use of (2.16). Now set

$$\overline{z}(y,t) = \begin{cases} \widetilde{z}(y,t) & if \quad y_N > 0, \\ -\widetilde{z}(y_1, \ldots y_{N-1}, -y_N, t) & if \quad y_N < 0. \end{cases}$$

Then $\overline{z}$ satisfies

$$\frac{\partial}{\partial t}\overline{z} - \operatorname{div}\left(\overline{H}\nabla\overline{z}\right) = \operatorname{div} \overline{g} + \overline{g}_0 \qquad in \qquad \Lambda \times (-\tau, T], \tag{2.18}$$

where

$$\overline{g}_0 = \begin{cases} g_0(y,t) & if \quad y_N > 0, \\ -g(y_1, \ldots, y_{N-1}, -y_N, t) & if \quad y_N < 0, \end{cases}$$

$$\overline{H} = \begin{cases} \left(H^{-1}\right)^T H^{-1} & if \quad y_N > 0, \\ \overline{I}\left(\left(H^{-1}\right)^T H^{-1}\right)(y_1, \ldots, y_{N-1}, -y_N)\,\overline{I} & if \quad y_N < 0, \end{cases}$$

$$\overline{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix},$$

$$\overline{g} = \begin{cases} g & if \quad y_N > 0, \\ -\overline{I}g(y_1, \ldots, y_{N-1}, -y_N, t) & if \quad y_N < 0. \end{cases}$$

Observe that $\overline{H}$ is still positive definite. In fact, in view of (4.6) in [SW], we can choose g_i so that $\overline{H}$ is continuous in Λ. We are ready to conclude from our earlier argument that

$$\fint_{Q_\rho(y_0,t_0)} \left(\overline{z} - (\overline{z})_\rho\right)^2 dydt \le c(A_0) \tag{2.19}$$

for all (y_0, t_0) such that $|y_0| \le \dfrac{A_0}{2}$, $t_0 \in (0, T]$, and $0 < \rho \le \dfrac{A_0}{2}$. It is easy to see that

$$\text{meas } \mathcal{T}\left(B_\rho(y_0)\right) = \text{meas } B_\rho(y_0), \tag{2.20}$$

$$B_{2\rho}\left(\mathcal{T}^{-1}(x_1)\right) \supset \mathcal{T}^{-1}\left(B_\rho(x_1)\right) \tag{2.21}$$

for all $x_1 \in U$, $\rho > 0$ with $B_\rho(x_1) \subset U$. For $(x_1, t_1) \in \mathcal{T}\left(\Lambda^+ \cap B_{\frac{A_0}{2}}(0)\right) \times (0, T]$ such that $2R_0(x_1, t_1) \le \dfrac{A_0}{2}$, we compute, with the aid of (2.19), (2.20), and (2.21), that

$$\begin{aligned} \fint_{Q_{R_0}(x_1,t_1)} (z - z_\rho)^2 dxdt &= c\frac{1}{R_0^{N+2}} \int_{t_1 - R_0^2}^{t_1} \int_{\mathcal{T}^{-1}\left(B_{R_0}(x_1)\right)} \left(\widetilde{z} - (\widetilde{z})_\rho\right)^2 dydt \\ &\le c \fint_{Q_{2R_0}(\mathcal{T}^{-1}(x_1), t_1)} \left(\overline{z} - (\overline{z})_{2R_0}\right)^2 dydt \\ &\le c. \end{aligned}$$

This completes the proof of (2.5). Hence Theorem 2 follows. ☐

Theorem 7. *Let σ, u^*, $\overline{\varphi}$ be given as before. Then any bounded capacity solution (u, φ) to* (1.1) *satisfies*

(P1) $u \in C^{\alpha, \frac{\alpha}{2}}\left(\overline{Q}_T\right)$ *for some* $\alpha \in (0, 1)$*;*

(P2) *for each* $q > 1$, $|\nabla u| \in L^q\left(0, T;\ W^{1,q}(\Omega)\right)$ *and* $|\nabla \varphi| \in L^\infty\left(0, T;\ W^{1,q}(\Omega)\right)$.

Since u is bounded, $\sigma(u)$ stays away from 0 below. Thus, this result is essentially the same as that in [YL]. The proof there is based upon an application of the single-layer potential theory. We claim that this result is a consequence of our earlier argument. To see this, note that for a.e. t in $(0,T)$, $\varphi(x,t)$ solves the boundary-value problem

$$\operatorname{div}(\sigma(u)\nabla\varphi) = 0 \qquad in \qquad \Omega, \tag{2.22a}$$

$$\varphi = \overline{\varphi} \qquad on \qquad \partial\Omega. \tag{2.22b}$$

Hence, the classical regularity theory for linear elliptic equations asserts

$$\operatorname{ess\,sup}_{(0,T)} \|\varphi(\cdot,t)\|_{C^\beta(\overline{\Omega})} \le c$$

for some $\beta \in (0,1)$. Set $\varphi_R(t) = ⨍_{\Omega_R} \varphi(y,t)\,dy$. Then we have

$$\|\varphi - \varphi_R\|_{\infty,P_R} \le cR^\beta.$$

Observe from (2.22a) that

$$\operatorname{div}(\sigma(u)\varphi\nabla\varphi) = \operatorname{div}(\sigma(u)(\varphi-\varphi_R)\nabla\varphi). \tag{2.23}$$

Thus, f in (2.7a) may be replaced by

$$f = \operatorname{div}(\sigma(u)(\varphi-\varphi_R)\nabla\varphi\chi_{Q_T} + \nabla u^*\chi_{Q_T}) - u_t^*\chi_{Q_T}$$

and $|v|_{D_R}$ in (2.8) may be estimated by

$$\begin{aligned} |v|_{D_R} &\le c\int_{P_R} \sigma(u)(\varphi-\varphi_R)^2 |\nabla\varphi|^2\,dxdt + CR^{N+2} \\ &\le CR^{N+2\beta} + cR^{N+2}. \end{aligned}$$

Consequently,

$$\int_{D_R} v^2 dxdt \le cR^{N+2+2\beta}.$$

So (2.13) now reads

$$\int_{Q_\rho} (z-z_\rho)^2\,dxdt \le cR^{N+2+2\beta} + c\left(\frac{\rho}{R}\right)^{N+2+2\alpha} \int_{Q_R} (z-z_R)^2\,dxdt$$

for all $0 < \rho \le R \le R_*$. Pick γ from $(0, \min\{\alpha,\beta\})$. Then we deduce from [G, p.86] that

$$\int_{Q_\rho} (z-z_\rho)^2\,dxdt \le c\rho^{N+2+2\gamma} + c\left(\frac{\rho}{R_*}\right)^{N+2+2\gamma} \int_{Q_{R_*}} (z-z_{R_*})^2\,dxdt$$

for all $0 < \rho \le R_*$. By chasing the proof in [G, pp.70–72], we obtain that z is locally Hölder continuous in $\overline{Q}_T\backslash\overline{S}_T$. The local Hölder continuity at the lateral boundary S_T can be obtained from (2.18). Once we know $\sigma(u)$ is continuous, a result in [R, p.82] states that $|\nabla\varphi| \in L^q(\Omega)$ for each $q > 1$ and a.e. t in $(0,T)$. In fact, we have

$$\operatorname{ess\,sup}_{(0,T)} \|\nabla\varphi\|_{q,\Omega} \le c.$$

This, in turn, implies

$$|\nabla u| \in L^q\left(0,T;\ W^{1,q}(\Omega)\right)$$

for each $q > 1$, because of a result in [BLP, pp. 273–274].

3. Partial regularity for u

The main result of this section is:

Theorem 8. *Assume (H1)–(H3),* (1.2)*, and* (1.3)*. Let (u, φ) be the solution constructed in* [X2]*. Then there exists a relatively open subset Q_0 of Q_T such that*

a. *u is locally Hölder continuous on Q_0;*
b. *$Q_T \backslash Q_0$ has an empty interior;*
c. *$u \in L^\infty(Q_T \backslash Q_0)$.*

The solution in [X2] is established as a limit of a sequence of approximate solutions. As we shall see, our proof of Theorem 8 relies on the approximation procedure employed in [X2]. Since classical weak solutions or capacity solutions are not known to be unique, we do not know if Theorem 8 holds for a solution obtained by other constructions. Note that the results in Section 2 hold for any capacity solutions.

We are not able to obtain the Hausdorff dimension of $Q_T \backslash Q_0$ due to the degeneracy involved.

Now consider a sequence of approximate problems:

$$\frac{\partial u_n}{\partial t} - \Delta u_n = \sigma_n(u_n)|\nabla\varphi_n|^2 \qquad in \qquad Q_T, \tag{3.1a}$$

$$\operatorname{div}(\sigma_n(u_n)\nabla\varphi_n) = 0 \qquad in \qquad Q_T, \tag{3.1b}$$

$$u_n = u^* \qquad on \qquad \partial pQ_T, \tag{3.1c}$$

$$\varphi_n = \overline{\varphi} \qquad on \qquad S_T \quad (n = 1, 2, \dots), \tag{3.1d}$$

where $\sigma_n(s) = \sigma(s) + \frac{1}{n}$. Then by virtue of a result in [X2] (u, φ) in Theorem 8 can be viewed as a strong limit of (u_n, φ_n) in $[L^2(Q_T)]^2$. For each fixed n, u_n satisfies the conclusions of Theorem 7. We shall proceed to derive a priori estimates for (u_n, φ_n).

Lemma 9. *There exists a positive constant c such that*

$$\|\varphi_n\|_{\infty, Q_T} \le c,$$

$$\int_{Q_T} \sigma_n(u_n)|\nabla\varphi_n|^2 \, dxdt \le c,$$

$$\operatorname*{ess\,sup}_{[0,T]} \int_\Omega u_n^2(x,t)\,dx + \int_{Q_T} |\nabla u_n|^2 \, dxdt \le c$$

for all n.

We refer the reader to [X2] for the proof.

Lemma 10. *For each $A > 0$,*

$$-\Delta\varphi_n = \operatorname{div}(F_A(u_n)\nabla\varphi_n) + G_A(u_n)\nabla\varphi_n\nabla u_n \qquad in \qquad \mathcal{D}'(\Omega) \tag{3.2}$$

for a.e. t in $(0, T)$, where

$$F_A(u_n) = \sigma_n(u_n)\frac{1}{A}\int_{u_n}^{u_n+A} \frac{1}{\sigma_n(s)}ds - 1, \tag{3.3}$$

$$G_A(u_n) = \frac{1}{A}\left(1 - \frac{\sigma_n(u_n)}{\sigma_n(u_n + A)}\right). \tag{3.4}$$

Proof. For each $A > 0$ and each $\xi \in C_0^\infty(\Omega)$, we have that

$$\xi \int_{u_n}^{u_n+A} \frac{1}{\sigma_n(s)} ds \in W_0^{1,2}(\Omega) \tag{3.5}$$

for each t in $[0,T]$. Observe from (3.1b) that

$$\operatorname{div}(\sigma_n(u_n)\nabla\varphi_n) = 0 \qquad in \qquad \mathcal{D}'(\Omega) \tag{3.6}$$

for a.e. t in $(0,T)$. We conclude that for a.e. t in $(0,T)$

$$\int_\Omega \sigma_n(u_n)\nabla\varphi_n \int_{u_n}^{u_n+A} \frac{1}{\sigma_n(s)} ds \nabla\xi dx + \int_\Omega \sigma_n(u_n)\nabla\varphi_n\xi\left(\frac{1}{\sigma_n(u_n+A)} - \frac{1}{\sigma_n(u_n)}\right)\nabla u_n dx = 0.$$

This yields the lemma. □

From now on, we will operate under (1.2) and (1.3). It is easy to see that

$$\left|\frac{\sigma_n(s+\tau)}{\sigma_n(s)} - 1\right| \le \left|\frac{\sigma(s+\tau)}{\sigma(s)} - 1\right| . \tag{3.7}$$

A lemma in [X2] indicates that if (1.2) holds, then for each $L > 0$ there exist $m_L > 0$, $M_L > 0$ with the property

$$m_L \le \frac{\sigma(s+\tau)}{\sigma(s)} \le M_L \qquad for \qquad (s,\tau) \in (-\infty,\infty)\times[-L,L] . \tag{3.8}$$

If this holds, then

$$m_L \le \frac{\sigma_n(s+\tau)}{\sigma_n(s)} \le M_L \qquad for \qquad (s,\tau) \in (-\infty,\infty)\times[-L,L] . \tag{3.9}$$

In this case one can show [X2] that $\int_{Q_T} |\nabla\varphi_n|^2 dxdt \le c$ for all n. Thus (u,φ) is a classical weak solution.

Theorem 11. *For each $q > 2$ there exists $c > 0$ such that*

$$\|\nabla\varphi_n\|_{q,Q_\rho(x,t)} \le c\left(\|\nabla u_n\|_{q,Q_R(x,t)} + \frac{R^{\frac{N+2}{q}}}{R-\rho}\right) \tag{3.10}$$

for all $(x,t) \in Q_T$ and $0 < \rho < R \le R_0(x,t)$.

Proof. Let $x_0 \in \Omega$ be given. Fix $R_*(x_0) > R > r > 0$. Choose $\xi \in C_0^\infty(B_R(x_0))$ so that

$$\xi \ge 0 \qquad on \qquad B_R \equiv B_R(x_0),$$
$$\xi = 1 \qquad on \qquad B_r,$$

and

$$|\nabla\xi| \le \frac{c}{R-r} .$$

Set

$$\psi_n = \varphi_n\xi .$$

A simple calculation yields

$$-\Delta\psi_n = \operatorname{div}\left(F_A\left(u_n\right)\nabla\psi_n\right) - \left(1 + F_A\left(u_n\right)\right)\nabla\varphi_n\nabla\xi + G_A\left(u_n\right)\nabla\psi_n\nabla u_n - G_A\left(u_n\right)\nabla\xi\varphi_n\nabla u_n - \operatorname{div}\left(\left(1 + F_A\left(u_n\right)\right)\varphi_n\nabla\xi\right) \quad in \quad \mathbf{R}^N \tag{3.11}$$

for a.e. t in $(0,T)$, where φ_n, u_n are understood to be zero outside B_R. Now for each $y \in \mathbf{R}^N$ and $\rho_0 > 0$ consider the problem

$$-\Delta\phi_n = 0 \quad in \quad B_{\rho_0}(y) \equiv B_{\rho_0}, \tag{3.12a}$$

$$\phi_n = \psi_n \quad on \quad \partial B_{\rho_0}. \tag{3.12b}$$

A result in [G, p.78] asserts that

$$\int_{B_\rho} \left|\nabla\phi_n - \left(\nabla\phi_n\right)_\rho\right|^2 dx \le c\left(\frac{\rho}{\rho_0}\right)^{N+2} \int_{B_{\rho_0}} \left|\nabla\phi_n - \left(\nabla\phi_n\right)_{\rho_0}\right|^2 dx \tag{3.13}$$

for all $0 < \rho \le \rho_0$, where $c > 0$ depends only on N. Subtract (3.12a) from (3.11), use $\psi_n - \phi_n$ as a test function in the resulting equation, and thereby obtain

(3.14)

$$\begin{aligned}
&\int_{B_{\rho_0}} \left|\nabla\left(\psi_n - \phi_n\right)\right|^2 dx \\
&\quad \le \int_{B_{\rho_0}} F_A^2\left(u_n\right)\left|\nabla\psi_n\right|^2 dx + c\int_{B_{\rho_0}} \left(1 + F_A\left(u_n\right)\right)\left|\nabla\varphi_n\right|\left|\nabla\xi\right| dx \\
&\quad + c\int_{B_{\rho_0}} G_A\left(u_n\right)\left|\nabla\psi_n\right|\left|\nabla u_n\right| dx + c\int_{B_{\rho_0}} G_A\left(u_n\right)\left|\nabla\xi\nabla u_n\right| dx \\
&\quad + c\int_{B_{\rho_0}} \left|\nabla\xi\right|^2\left(1 + F_A\left(u_n\right)\right)^2 dx.
\end{aligned}$$

Here, we used the fact that $\{\phi_n\}$, $\{\psi_n\}$, $\{\varphi_n\}$ are all bounded in $L^\infty\left(Q_T\right)$. Observe that there exists a θ_n in $(0,1)$ so that

$$\frac{1}{A}\int_{u_n}^{u_n+A} \frac{1}{\sigma_n(s)} ds = \frac{1}{\sigma_n\left(u_n + \theta_n A\right)}\ .$$

We can infer from (3.9) that

$$\left|1 + F_A\left(u_n\right)\right| \le c(A),\ \left|\ G_A\left(u_n\right)\right| \le c(A)$$

for some $c(A) > 0$. For any $\varepsilon > 0$, we conclude from (3.14) that

(3.15)

$$\begin{aligned}\int_{B_{\rho_0}} &|\nabla(\psi_n - \phi_n)|^2 dx \\ &\leq \left(\left\|F_A^2(u_n)\right\|_{\infty,B_R} + \varepsilon\right) M\left(|\nabla\psi_n|^2\right)\rho_0^N \\ &\quad + \rho_0^N\left(\frac{c(A)}{(R-r)}M(|\nabla\varphi_n|) + c(\varepsilon, A)\, M\left(|\nabla u_n|^2\right) + c(A)\, M\left(|\nabla\xi|^2\right)\right) \\ &\equiv D_n\rho_0^N + E_n\rho_0^N,\end{aligned}$$

where $M\left(|\nabla\psi_n|^2\right)$, $M\left(|\nabla u_n|^2\right)$, $M(|\nabla\varphi_n|)$, and $M\left(|\nabla\xi|^2\right)$ are the maximal functions of $|\nabla\psi_n|^2$, $|\nabla u_n|^2\chi_{B_R}$, $|\nabla\varphi_n|\chi_{B_R}$, and $|\nabla\xi|^2$, respectively. For $0 < \rho \leq \rho_0$, we estimate, with the aid of (3.13) and (3.15), that

$$\int_{B_\rho}\left(\nabla\psi_n - (\nabla\psi_n)_\rho\right)^2 dx \tag{3.16}$$

$$\begin{aligned} &\leq c\int_{B_\rho}(\nabla\psi_n - \nabla\phi_n)^2 dx + \int_{B_\rho}\left(\nabla\phi_n - (\nabla\phi_n)_\rho\right)^2 dx \\ &\leq c\int_{B_{\rho_0}}(\nabla\psi_n - \nabla\phi_n)^2 dx + c\left(\tfrac{\rho}{\rho_0}\right)^{N+2}\int_{B_{\rho_0}}\left(\nabla\psi_n - (\nabla\psi_n)_{\rho_0}\right)^2 dx \\ &\leq c(D_n + E_n)\rho_0^N + c\left(\tfrac{\rho}{\rho_0}\right)^{N+2}\int_{B_{\rho_0}}\left(\nabla\psi_n - (\nabla\psi_n)_{\rho_0}\right)^2 dx \\ &\leq c(D_n + E_n)\rho_0^N + c\left(\tfrac{\rho}{\rho_0}\right)^{N+2}\int_{B_{\rho_0}}|\nabla\psi_n|^2 dx.\end{aligned}$$

We are ready to employ an argument in [DM]. For this purpose, we rewrite (3.16) as

$$\begin{aligned}&⨍_{B_\rho}\left(\nabla\psi_n - (\nabla\psi_n)_\rho\right)^2 dx \\ &\quad\leq c\left[\left(\left\|F_A^2(u_n)\right\|_{\infty,B_R} + \varepsilon\right)\left(\frac{\rho_0}{\rho}\right)^N + \left(\frac{\rho}{\rho_0}\right)^2\right] M\left(|\nabla\psi_n|^2\right) + c\left(\frac{\rho_0}{\rho}\right)^N E_n\ .\end{aligned}$$

Remember that

$$|F_A(u_n)| \leq \left|\frac{\sigma(u_n)}{\sigma(u_n + \theta_n A)} - 1\right| \to 0$$

uniformly as $A \to 0$. Thus, for each $\tau \in (0,1)$, we may choose A and ε so that

$$⨍_{B_{\rho_0\tau}}\left(\nabla\psi_n - (\nabla\psi_n)_{\rho_0\tau}\right)^2 dx \leq c\tau^2 M\left(|\nabla\psi_n|^2\right) + c(\tau)\, E_n\ .$$

Since ρ_0 is arbitrary, we can conclude that

$$\left((\nabla\psi_n)^\sharp\right)^2 \leq c\tau^2 M\left(|\nabla\psi_n|^2\right) + c(\tau)\, E_n\ , \tag{3.17}$$

where $(\nabla\psi_n)^\sharp$ is the sharp maximal function associated with $\nabla\psi_n$. Remember that $u_n \in W^{1,q}(\Omega)$ for a.e. t in $(0,T)$ and each $q > 2$. Now we are in a position to apply the Fefferman-Stein inequality in [FS]. This results in

$$\begin{aligned}\int_{B_R} |\nabla\psi_n|^q\,dx &= \int_{\mathbf{R}^N} |\nabla\psi_n|^q\,dx \\ &\le \int_{\mathbf{R}^N} (M(\nabla\psi_n))^q\,dx \\ &\le c(q)\int_{\mathbf{R}^N} \left[(\nabla\psi_n)^\sharp\right]^q dx \\ &\le c(q)\,\tau^q \int_{\mathbf{R}^N} \left(M\left(|\nabla\psi_n|^2\right)\right)^{\frac{q}{2}} dx + c(\tau,q)\,E_n^{\frac{q}{2}}\end{aligned}$$

for each $q > 2$. Recalling the definition of E_n and the Hardy-Littlewood maximal theorem [DM], we arrive at

(3.18)
$$\begin{aligned}\int_{\mathbf{R}^N} |\nabla\psi_n|^q\,dx \le c(q)\,\tau^q \int_{\mathbf{R}^N} |\nabla\psi_n|^q\,dx + \frac{c(q)}{(R-r)^{\frac{q}{2}}}\int_{B_R} |\nabla\varphi_n|^{\frac{q}{2}}\,dx \\ + c(q)\int_{B_R} |\nabla u_n|^q\,dx + c(q)\int_{\mathbf{R}^N} |\nabla\xi|^q\,dx.\end{aligned}$$

Now choose τ so that $c(q)\,\tau^q < \dfrac{1}{2}$. We derive from (3.18) that

(3.19)
$$\begin{aligned}\int_{B_r} |\nabla\varphi_n|^q\,dx &\le \int_{\mathbf{R}^N} |\nabla\psi_n|^q\,dx \\ &\le \frac{c}{(R-r)^{\frac{q}{2}}}\int_{B_R} |\nabla\varphi_n|^{\frac{q}{2}}\,dx + c\int_{B_R} |\nabla u_n|^q\,dx + c\frac{R^N}{(R-r)^q} \\ &\le \varepsilon\int_{B_R} |\nabla\varphi_n|^q\,dx + c\int_{B_R} |\nabla u_n|^q\,dx + \frac{c(\varepsilon)\,R^N}{(R-r)^q}\end{aligned}$$

for each $\varepsilon > 0$. For $k \in \{1,2,\dots\}$, define

$$\rho_k = R - \frac{\theta R}{2^k}\ ,$$

where $\theta \in (0,1)$. Set $r = \rho_k$, $R = \rho_{k+1}$ in (3.19) to obtain

$$\begin{aligned}\int_{B_{\rho_k}} |\nabla\varphi_n|^q\,dx \le \varepsilon\int_{B_{\rho_{k+1}}} |\nabla\varphi_n|^q\,dx + c\int_{B_R} |\nabla u_n|^q\,dx \\ + \frac{c(\varepsilon)\,R^{N-q}2^{qk}}{\theta^q}\ .\end{aligned}$$

In view of a result in [D, p.13], we derive, by suitably selecting ε, that

$$\int_{R(1-\theta)} |\nabla \varphi_n|^q \, dx \le c\,(q) \int_{B_R} |\nabla u_n|^q \, dx + c\,(q)\, \frac{R^N}{(R\theta)^q} \ .$$

A simple integration yields (3.10). □

Proof of Theorem 8. Fix (x_0, t_0) in Q_T. Let $R > 0$ be such that $Q_R\,(x_0, t_0) \equiv Q_R \subset Q_T$. Consider the problem

$$\frac{\partial v_n}{\partial t} - \Delta v_n = \operatorname{div}\left(\sigma_n\,(u_n)\,\varphi_n \nabla \varphi_n\right) \qquad in \qquad Q_R \ , \tag{3.20a}$$

$$v_n = 0 \qquad on \qquad \partial p Q_R \ . \tag{3.20b}$$

In light of a result in [BLP, pp.273–274], we have that for each $q > 2$ there exists a $c\,(q)$ independent of R such that

$$\|\nabla v_n\|_{q,Q_R} \le c\,(q)\, \|\sigma_n\,(u_n)\,\varphi_n \nabla \varphi_n\|_{q,Q_R} \ . \tag{3.21}$$

Set $w_n = u_n - v_n$. Then w_n satisfies

$$\frac{\partial w_n}{\partial t} - \Delta w_n = 0 \qquad in \qquad Q_R \ , \tag{3.22a}$$

$$w_n = u_n \qquad on \qquad \partial p Q_R \ . \tag{3.22b}$$

We infer from a result in [D] that

$$\|\nabla w_n\|_{\infty, Q_\rho} \le \frac{c}{(R-\rho)^{\frac{N+2}{2}}} \left(\int_{Q_R} |\nabla w_n|^2 \, dxdt \right)^{\frac{1}{2}}$$

for each $0 < \rho < R$. Keeping these in mind, we calculate, for $\rho < R$, that

(3.23)

$$\begin{aligned}
\|\nabla u_n\|_{q,Q_\rho} &\le \|\nabla v_n\|_{q,Q_R} + \|\nabla w_n\|_{q,Q_\rho} \\
&\le \|\nabla v_n\|_{q,Q_R} + \rho^{\frac{N+2}{q}} \|\nabla w_n\|_{\infty,Q_\rho} \\
&\le c\,(q)\, \|\sigma_n\,(u_n)\,\varphi_n \nabla \varphi_n\|_{q,Q_R} \\
&\quad + \frac{c\rho^{\frac{N+2}{q}}}{(R-\rho)^{\frac{N+2}{2}}} \left(\|\nabla u_n\|_{2,Q_R} + \|\nabla v_n\|_{2,Q_R} \right) \\
&\le c\,(q)\, \|\sigma_n\,(u_n)\,\varphi_n \nabla \varphi_n\|_{q,Q_R} \\
&\quad + \frac{c\rho^{\frac{N+2}{q}}}{(R-\rho)^{\frac{N+2}{2}}} \left(\|\nabla u_n\|_{2,Q_R} + \|\sigma_n\,(u_n)\,\varphi_n \nabla \varphi_n\|_{2,Q_R} \right).
\end{aligned}$$

Define, for $k \in \{0, 1, 2, \dots\}$,

$$\rho_k = R - \frac{R}{2^{k+1}} \ .$$

Set $\rho = \rho_k$, $R = \rho_k + \frac{1}{2}\,(\rho_{k+1} - \rho_k) = \frac{1}{2}\,(\rho_{k+1} + \rho_k)$ in (3.10) to obtain

$$\|\nabla \varphi_n\|_{q,Q_\rho} \le c\,(q) \left(\|\nabla u_n\|_{q,Q_{\frac{1}{2}(\rho_{k+1}+\rho_k)}} + cR^{\frac{N+2}{q}-1} 2^k \right). \tag{3.24}$$

Let $\rho = \frac{1}{2}(\rho_{k+1}+\rho_k)$, $R=\rho_{k+1}$ in (3.23) to get

$$\|\nabla u_n\|_{q,Q_{\frac{1}{2}(\rho_{k+1}+\rho_k)}} \le c(q)\|\sigma_n(u_n)\|_{\infty,Q_R}\|\nabla\varphi_n\|_{q,Q_{\rho_{k+1}}} + cR^{\frac{N+2}{q}-\frac{N+2}{2}}\left(2^{\frac{N+2}{2}}\right)^k.$$

Use this in (3.24) to obtain

$$\|\nabla\varphi_n\|_{q,Q_{\rho_k}} \le c(q)\|\sigma_n(u_n)\|_{\infty,Q_R}\|\nabla\varphi_n\|_{q,Q_{\rho_{k+1}}} + c\left(R^{\frac{N+2}{q}-1}+R^{\frac{N+2}{q}-\frac{N+2}{2}}\right)\left(2^{\frac{N+2}{2}}\right)^k. \tag{3.25}$$

Thus, if

$$c(q)\|\sigma_n(u_n)\|_{\infty,Q_R}\cdot 2^{\frac{N+2}{2}} \le \frac{1}{2}\,, \tag{3.26}$$

we have

$$\|\nabla\varphi_n\|_{q,Q_{\frac{R}{2}}} \le c\left(R^{\frac{N+2}{q}-1}+R^{\frac{N+2}{q}-\frac{N+2}{2}}\right). \tag{3.27}$$

Recall that $\lim_{s\to\infty}\sigma(s)=0$ and $\sigma_n(s)=\sigma(s)+\frac{1}{n}$. We conclude that there is an $M_0>0$ such that (3.26) holds whenever n is large enough and

$$\operatorname*{ess\,inf}_{Q_R} u_n \ge M_0. \tag{3.28}$$

Now let $(x_0,t_0)\in Q_T$ and c^* be given as in (2.1). Assume that there exists an $R_1\in(0,R_0(x_0,t_0)/2)$ such that

$$⨍_{Q_{2R_1}(x_0,t_0)} u\,dxdt > c^*+M_0.$$

Then

$$⨍_{Q_{2R_1}(x_0,t_0)} u_n\,dxdt > c^*+M_0$$

for n sufficiently large, since a result in [X2] asserts that $u_n\to u$ strongly in $L^2(Q_T)$. By virtue of Theorem 2,

$$\begin{aligned}\operatorname*{ess\,inf}_{Q_{R_1}} u_n &\ge ⨍_{Q_{2R_1}} u_n\,dxdt - c^*\\ &> M_0 \quad\text{for } n \text{ sufficiently large.}\end{aligned}$$

This means that (3.27) holds for $R=R_1$ and n large enough. Thus,

$$\|\nabla\varphi\|_{q,Q_{\frac{R_1}{2}}} \le c(q,R_1) \tag{3.29}$$

for all $q>2$. Define

$$\Gamma = \left\{(x,t)\in Q_T:\ \exists R\in(0,R_0(x,t)/2): ⨍_{Q_R(x,t)} u\,dxdt > c^*+M_0\right\}.$$

Clearly, Γ is open. For each $(x,t)\in\Gamma$, there is a $\rho_0>0$ such that u is Hölder continuous on $Q_{\rho_0}(x,t)$ due to (3.29). We see that

$$u \le c^*+M_0$$

a.e. on $Q_T\backslash\Gamma$. Now, let

$$Q_0 = \Gamma \cup \text{Interior of} \quad Q_T\backslash\Gamma.$$

The local Hölder continuity of u on the interior of $Q_T\backslash\Gamma$ is due to a local version of Theorem 7. This completes the proof. □

References

[AC] S.N. Antontsev and M. Chipot, The thermistor problem : existence, smoothness, uniqueness, and blow up, *SIAM J. Math. Anal.* **25** (1994), 1128–1156. MR **95f:**35013

[BLP] A. Bensoussan, J.L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland Publ. Co., Amsterdam, 1978. MR **82h:**35001

[C] X. Chen, Existence and regularity of solutions of a nonlinear degenerate elliptic system arising from a thermistor problem, *J. Partial Differential Equations* **7** (1994), 19–34. MR **95c:**35078

[D] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, 1993. MR **94h:**35130

[DM] E. DiBenedetto and J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, *Amer. J. Math.* **115** (1993), 1107–1134. MR **94i:**35077

[FS] C. Fefferman and E.M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193. MR **56:**6263

[G] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, Princeton, New Jersey, 1983. MR **86b:**49003

[LSU] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural′ceva, *Linear and Quasilinear Equations of Parabolic Type*, *Transl. Math. Mono.*, Vol. **23**, A.M.S., Providence, Rhode Island, 1968. MR **39:**3159

[K] R.V. Kohn, The method of partial regularity as applied to the Navier-Stokes equations, *Seminar on Nonlinear Partial Differential Equations*, S.S. Chern, Ed., Springer-Verlag, New York, 1984, 117–128. MR **85g:**35005

[R] J.F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland Math. Studies, Vol. **134**, North-Holland, Amsterdam, 1987. MR **88d:**35006

[SSX] P. Shi, M. Shillor, and X. Xu, Existence of a solution to the Stefan problem with Joule's heating, *J. Differential Equations* **105** (1993), 239–263. MR **94k:**35338

[SW] P. Shi and S. Wright, Higher integrability of the gradient in linear elasticity, *Math. Ann.* **299** (1994), 435–448. MR **95f:**35034

[XA] H. Xie and W. Allegretto, $C^\alpha(\overline{\Omega})$-solution of a class of nonlinear degenerate elliptic systems arising in the thermistor problem, *SIAM J. Math. Anal.* **22**(1991), 1491–1499. MR **92m:**35099

[X1] X. Xu, A strongly degenerate system involving an equation of parabolic type and an equation of elliptic type, *Comm. Partial Differential Equations* **18** (1993), 199–213. MR **95c:**35182

[X2] X. Xu, A p-Laplacian problem in L^1 with nonlinear boundary conditions, *Comm. Partial Differential Equations* **19** (1994), 143–176. MR **95b:**35075

[Y] G. Yuan, Regularity of solutions of the nonstationary thermistor problem, *Applicable Anal.* **53**(1994), 149–169. CMP 1996:9

[YL] G. Yuan and Z. Liu, Existence and uniqueness of the C^α-solution for the thermistor problem with mixed boundary value, *SIAM J. Math. Anal.* **25** (1994), 1157–1166. MR **95f:**35097

Department of Mathematics and Statistics, Mississippi State University, Mississippi State, Mississippi 39762

E-mail address: xxu@math.msstate.edu